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Correction to Hawking Pure Thermal Spectrum of Stationary Kaluza-Klein Black Hole

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Applying Parikh's quantum tunneling method, the tunneling characteristics of stationary Kaluza-Klein black hole is researched. The result shows that the tunneling rate across the event horizon of the black hole is relevant to the change of Bekenstein-Hawking entropy and the derived radiation spectrum deviates from pure thermal when the self-gravitation, energy conservation and angular momentum conservation are taken into consideration. Finally, we use the obtained results to reduce to stationary Kerr black hole and static Swarzschild black hole, and find that only ignoring the spectrum at higher energies the tunneling radiation spectrum is consistent with Hawking pure thermal one.

KEY WORDS: Kaluza-Klein black hole; energy conservation; angular momentum conservation; tunneling rate; Bekenstein-Hawking entropy.

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1. INTRODUCTION

In 1970s, Hawking discovered and proved the thermal radiation of black hole, which is greatly meaningful to research the evolution of the fixed stars (Hawking, 1975). Since then, black hole thermodynamic properties have been researched by a lot of people. There are two methods to research Hawking radiation, namely quantum field theory method and Damour-Ruffini method (Damour and Ruffini, 1976). In the past few decades, people have applied the two methods to carry on a series of research on Hawking radiation of static, stationary and non-stationary black holes (Zhang and Zhao, 2002; Liu and Xu, 2002; Jiang *et al.*, 2005; Wu and Cai, 2000; Xu, 1982; Yang and Lin, 2001). But the similarities between the two methods are that the space-time background is fixed and the derived thermal spectrum is pure thermal. So in the process of solving Hawking thermal spectrum, there are two points worth discussing: first, the information lost, which means the

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pure quantum state will be reduced to the mixture, using the language of quantum field theory, the ingoing state is pure and the outgoing is mixture, which disobeys the underling unitary theory. Secondly, the technical problem, we have learned that black hole radiation is the result of quantum tunneling effect at present, but till now, the causes of the tunneling barrier are indistinct for us. The related references do not use the language of the quantum tunneling to discuss Hawking thermal radiation, so strictly speaking, it is not the quantum tunneling method.

Recently, Parikh has present a quantum tunneling model and carried out a research on the tunneling radiation of Schwarzschild black hole and Reissner-Nordström black hole by using a coordinate system well-behaved at the event horizon (Parikh, 2004; Parikh and Wiltzek, 2000; Parikh, 2004). The result shows the radiation spectrum is not pure thermal under the consideration of the energy conservation and the background spacetime unfixed and the tunneling barrier is the result of the particle's self-gravitation. Thus we can get a new method to study Hawking radiation. The paper advances Parikh's method and discusses the tunneling characteristics of stationary Kaluza-Klein black hole. Due to the angular speed of the black hole $\Omega \neq 0$, the discussion is different from Parikh's. This is a subject that is worth studying but not be studied at present. Through calculating the tunneling rate, we derive the corrected spectrum of Kaluza-Klein black hole. The result indicates that Hawking radiation spectrum is not strictly exact one of the black hole, considering energy conservation, angular momentum conservation and self-gravitation, the tunneling radiation spectrum is connected with the change of Bekenstein-Hawking entropy and acts as a correction to Hawking radiation spectrum. Finally, we use the obtained results to reduce to stationary Kerr black hole and static Swarzschild black hole. The outline of the paper is organized as follows. In Section 2, we give out the event horizon and infinite red-shift surface of Kaluza-Klein black hole; In Section 3, Hawking pure thermal spectrum in the dragging coordinate system from Klein-Gordon equation is researched; Subsequently, in order to eliminate the coordinate singularity and make space in radial flat Eulidean to constant-time slices, we introduce general Painlevé coordinate transformation; In Section 5, we discuss the tunneling radiation characteristics of Kaluza-Klein black hole; Finally, in special cases, we reduce the results to stationary Kerr black hole and static Swarzschild black hole and obtain that the tunneling radiation spectrum is truly exact, only ignoring the spectrum at higher energies, and the tunneling radiation spectrum is consistent with Hawking pure thermal one.

2. THE EVENT HORIZON AND INFINITE RED-SURFACE OF KALUZA-KLEIN BLACK HOLE

According to Wang (2004), the line element of Kaluza-Klein black hole can be written in the following form

$$ds^2 = -\frac{1-Z}{B}dt_{kk}^2 - \frac{2aZ\,\sin^2\theta}{B\sqrt{1-\nu^2}}dt_k d\varphi$$
$$+\left[B(r^2+a^2)+a^2\,\sin^2\theta\frac{Z}{B}\right]\sin^2\theta d\varphi^2 + \frac{B\Sigma}{\Delta_r}dr^2 + B\Sigma d\theta^2 \tag{1}$$

where

$$Z = \frac{2mr}{\Sigma}, B = \left(1+\frac{\nu^2 Z}{1-\nu^2}\right)^{1/2} \Sigma = r^2 + a^2\,\cos^2\theta, \Delta_r = r^2+a^2-2mr. \tag{2}$$

t_{kk} is the coordinate time of Kaluza-Klein black hole, a and ν are the angular momentum of the unit mass and the velocity respectively. The total mass M, charge Q and angular momentum J_m of the black hole are given by

$$M = m\left[1+\frac{\nu^2}{2(1-\nu^2)}\right], \quad Q = \frac{m\nu}{1-\nu^2}, \quad J_m = \frac{ma}{\sqrt{1-\nu^2}}. \tag{3}$$

From the null super-surface equation

$$g^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial f}{\partial x^\nu} = 0, \tag{4}$$

we can get the event horizons of the black hole

$$r_\pm = m \pm \sqrt{m^2-a^2} = \frac{2(1-\nu^2)}{2-\nu^2}M \pm \sqrt{\left[\frac{2(1-\nu^2)}{2-\nu^2}\right]^2 M^2 - a^2}. \tag{5}$$

We demand a constant-time slice and $r = r_+$ in Eq. (1), the new line element can be written

$$d\sigma^2 = B(r_+^2 + a^2\cos^2\theta)d\theta^2 + \left[B(r_+^2+a^2)+a^2\sin^2\theta\frac{Z}{B}\right]\sin^2\theta d\varphi^2, \tag{6}$$

the determinant of the above two dimensional line element is

$$g = \frac{1}{1-\nu^2}(r_+^2+a^2)\sin^2\theta. \tag{7}$$

So the surface area of the black hole can be expressed as

$$A_+ = \int\sqrt{g}d\theta d\varphi = \frac{4\pi}{\sqrt{1-\nu^2}}(r_+^2+a^2)$$
$$= \frac{8\pi}{\sqrt{1-\nu^2}}\left\{\left[\frac{2(1-\nu^2)}{2-\nu^2}\right]^2 M^2 + \frac{2(1-\nu^2)}{2-\nu^2}M\sqrt{\left[\frac{2(1-\nu^2)}{2-\nu^2}\right]^2 M^2 - a^2}\right\}. \tag{8}$$

The infinite red-shift surfaces are given by the equation $g_{00} = 0$, so we can get the infinite red-shift surfaces

$$r_{\pm}^{s} = m \pm \sqrt{m^2 - a^2\cos^2\theta} = \frac{2(1-\nu^2)}{2-\nu^2}M \pm \sqrt{\left[\frac{2(1-\nu^2)}{2-\nu^2}\right]^2 M^2 - a^2\cos^2\theta}. \tag{9}$$

Obviously, the event horizons and the infinite red-shift surfaces are not coincident with each other, and there is a energy layer between them. In order to eliminate it, making the dragging coordinate transformation

$$\dot{\varphi} = \frac{d\varphi}{dt_{kk}} = -\frac{g_{03}}{g_{33}}, \tag{10}$$

the line element (1) can be transformed as follows

$$ds^2 = \hat{g}_{00}dt_{\mathrm{kk}}^2 + \frac{B\Sigma}{\Delta_r}dr^2 + B\Sigma d\theta^2, \tag{11}$$

where

$$\hat{g}_{00} = g_{00} - \frac{g_{03}^2}{g_{33}} = -\frac{B\Sigma\Delta_r(1-\nu^2)}{(r^2+a^2)^2 - \Delta_r(a^2\sin^2\theta + \nu^2\Sigma)}. \tag{12}$$

In fact, the line element (11) represents a 3-dimensional hyper-surface in 4-dimensional Kaluza-Klein space-time. So the event horizon and the infinite red-shift surface in the dragging coordinate system satisfy

$$r_{\pm} = r_{\pm}^{s} = \frac{2(1-\nu^2)}{2-\nu^2}M \pm \sqrt{\left[\frac{2(1-\nu^2)}{2-\nu^2}\right]^2 M^2 - a^2} \tag{13}$$

so, the infinite red-shift surface and the event horizon in the dragging coordinate system are coincident with each other.

3. HAWKING PURE THERMAL SPECTRUM OF KALUZA-KLEIN BLACK HOLE IN THE DRAGGING COORDINATE SYSTEM

Now, Let us move on to discuss Hawking pure thermal spectrum in the dragging coordinate system from Klein-Gordon equation. From Eq. (11), we can obtain the components of non-null inverse metric tensor

$$g = -\frac{B^3\Sigma^3(1-\nu^2)}{(r^2+a^2)^2 - \Delta_r(a^2\sin^2\theta + \nu^2\Sigma)}, \quad g^{11} = \frac{\Delta_r}{B\Sigma}, \quad g^{22} = \frac{1}{B\Sigma}$$
$$\hat{g}^{00} = g^{00} = -\frac{(r^2+a^2)^2 - \Delta_r(a^2\sin^2\theta + \nu^2\Sigma)}{B\Sigma\Delta_r(1-\nu^2)}. \tag{14}$$

For the sake of simplicity, we consider Hawking radiation of the uncharged scalar particles. In the curved space-time, Klein-Gordon equation can be expressed as

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^{\nu}}\Phi\right)-u^{2}\Phi=0, \tag{15}$$

substituting Eq. (14) into Eq. (15), we can obtain

$$\hat{g}^{00}\frac{\partial^{2}\Phi}{\partial t_{kk}^{2}}+g^{11}\frac{\partial^{2}\Phi}{\partial r^{2}}+\frac{1}{\sqrt{-g}}\frac{\partial\Phi}{\partial r}\frac{\partial}{\partial r}(\sqrt{-g}g^{11}) \\ +g^{22}\frac{\partial^{2}\Phi}{\partial\theta^{2}}+\frac{1}{\sqrt{-g}}\frac{\partial\Phi}{\partial\theta}\frac{\partial}{\partial\theta}(\sqrt{-g}g^{22})-u^{2}\Phi=0. \tag{16}$$

Carrying on the separation variable to Eq. (16) in the following form

$$\Phi=R(r)\Theta(\theta)e^{i\varsigma\varphi-i\omega t_{\mathrm{kk}}}, \tag{17}$$

and considering the dragging coordinate transformation (10), we can obtain the following expression (Zhang and Zhao, 2005)

$$g^{11}\frac{d^{2}R(r)}{dr^{2}}+\frac{R(r)}{\Theta(\theta)}\left[g^{22}\frac{d^{2}\Theta(\theta)}{d\theta^{2}}+\frac{1}{\sqrt{-g}}\frac{\partial}{\partial\theta}(\sqrt{-g}g^{22})\frac{d\Theta(\theta)}{d\theta}\right] \\ +\frac{1}{\sqrt{-g}}\frac{\partial}{\partial r}(\sqrt{-g}g^{11})\frac{dR(r)}{dr}=\left[u^{2}+\left(\omega+\varsigma\frac{g_{03}}{g_{33}}\right)^{2}g^{00}\right]R(r). \tag{18}$$

Introducing the tortoise coordinate transformation

$$r_{*}=\frac{1}{2\kappa}\mathrm{In}(r-r_{+}) \tag{19}$$

where $\kappa=\frac{(r_{+}-r_{-})\sqrt{1-v^{2}}}{2(r_{+}^{2}+a^{2})}$, is the surface gravity of the event horizon, Eq. (18) will be transformed into the following form

$$\frac{d^{2}R(r)}{dr_{*}^{2}}-2\kappa\frac{dR(r)}{dr_{*}}+2\kappa\left(r-r_{+}\right)\left[\frac{1}{\sqrt{-g}}\frac{\partial}{\partial r}(\sqrt{-g})+\frac{1}{g^{11}}\frac{\partial g^{11}}{\partial r}\right]\frac{dR\left(r\right)}{dr_{*}} \\ +\frac{4\kappa^{2}\left(r-r_{+}\right)^{2}}{g^{11}}\frac{R\left(r\right)}{\Theta\left(\theta\right)}\times\left[g^{22}\frac{d^{2}\Theta\left(\theta\right)}{d\theta^{2}}+\frac{1}{\sqrt{-g}}\frac{\partial}{\partial\theta}(\sqrt{-g}g^{22})\frac{d\Theta\left(\theta\right)}{d\theta}\right] \\ =\frac{4\kappa^{2}\left(r-r_{+}\right)^{2}}{g^{11}}\left[u^{2}+\left(\omega+\varsigma\frac{g_{03}}{g_{33}}\right)^{2}g^{00}\right]R\left(r\right). \tag{20}$$

In the vicinity of the event horizon, namely $r \to r_+$, we have

$$\frac{4\kappa^2 (r - r_+)^2}{g^{11}} \left[u^2 + \left(\omega + \varsigma \frac{g_{03}}{g_{33}} \right)^2 g^{00} \right] R(r) = -(\omega - \varsigma \Omega_+)^2 R(r), \quad (21)$$

substituting Eq. (21) into Eq. (20), we can get the standard wave equation near the stationary Kaluza-Klein black hole

$$\frac{d^2 R(r)}{dr_*^2} + (\omega - \varsigma \Omega_+)^2 R(r) = 0, \quad (22)$$

where Ω_+ is the dragging angular velocity at the event horizon. Solving Eq. (22) can we obtain the radial wave function of uncharged particles ingoing and outgoing the stationary Kaluza-Klein black hole

$$\Phi_{\text{in}} = e^{-i\omega v}, \quad \Phi_{\text{out}} = e^{-i\omega v} e^{2i(\omega - \omega_0) r_*}, \quad (23)$$

where $v = t_{\text{kk}} + \frac{\omega - \omega_0}{\omega} r_*$ is the advanced Eddington-Finkelstein coordinate. Φ_{out} can be written as follows near the event horizon

$$\Phi_{\text{out}} = e^{-i\omega v} (r - r_+)^{i(\omega - \omega_0)/\kappa}, \quad (24)$$

so Φ_{in} is analytical on the event horizon, while Φ_{out} has a logarithm singularity. By analytical continuation rotating $-\pi$ through the lower-half complex r-plane

$$(r \to r_+) \to |r - r_+| e^{-i\pi} = (r_+ - r) e^{-i\pi}, \quad (25)$$

and using the Damour-Ruffini stretch method of analysis, and extending it to the inside of the event horizon, we can get the spectrum of the Hawking radiation

$$N_\omega = \frac{1}{e^{(\omega - \varsigma \Omega_+)/T} - 1} = \frac{1}{e^{\xi A_+} - 1} \approx e^{-\xi A_+}, \quad (26)$$

where A_+ is the surface area of the black hole, and

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi} \frac{\sqrt{1 - v^2}\sqrt{[2(1 - v^2)/(2 - v^2)]^2 M^2 - a^2}}{[2(1 - v^2)/(2 - v^2)]^2 M^2 + [2(1 - v^2)/(2 - v^2)] M \sqrt{[2(1 - v^2)/(2 - v^2)]^2 M^2 - a^2}}$$

$$\xi = \frac{(\omega - \varsigma \Omega_+)}{2\sqrt{[2(1 - v^2)/(2 - v^2)]^2 M^2 - a^2}}. \quad (27)$$

From Eq. (26), we can learn that Hawking radiation spectrum can also be obtained in the dragging coordinate system, and the fixed space-time background results in the derived pure thermal spectrum. In fact, the mass of the black hole varies with its emission, and the event horizon also changes under the consideration of energy conservation and angular momentum conservation, which leads to the space-time background unfixed. In the following section, we will discuss Hawking tunneling radiation in the dragging coordinate system.

4. PAINLEVÉ-KALUZA-KLEIN COORDINATES

It is necessary to eliminate the coordinate singularity to analyze the Hawking tunneling radiation. From Eq. (11), the new line element in the dragging coordinate system still exists the coordinate singularity at the event horizon and it is not flat Eulidean space in radial to constant-time slices. So we further perform general Painlevé coordinate transformation (Zhang and Zhao, 2005; Painleve and Hebd, 1921)

$$dt_{\text{kk}} = dt + F\left(r, \theta\right) dr + G\left(r, \theta\right) d\theta, \tag{28}$$

where $F\left(r, \theta\right)$ and $G\left(r, \theta\right)$ are two to be determined functions of r and θ, and the integrability condition of Eq. (28) is

$$\frac{\partial F\left(r, \theta\right)}{\partial \theta} = \frac{\partial G\left(r, \theta\right)}{\partial r}, \tag{29}$$

Substituting Eq. (28) into Eq. (11), we have

$$\begin{aligned} ds^2 = {} & \hat{g}_{00} dt^2 + 2\hat{g}_{00} F\left(r, \theta\right) dt dr + [\hat{g}_{00} F^2\left(r, \theta\right) + B\Sigma\Delta_r^{-1}] dr^2 \\ & + [\hat{g}_{00} G^2\left(r, \theta\right) + B\Sigma] d\theta^2 \\ & + 2\hat{g}_{00} F\left(r, \theta\right) G\left(r, \theta\right) dr d\theta + 2\hat{g}_{00} G\left(r, \theta\right) dt d\theta, \end{aligned} \tag{30}$$

where the new time coordinate can be expressed as

$$t = t_{\text{kk}} - \int F\left(r, \theta\right) dr + G\left(r, \theta\right) d\theta. \tag{31}$$

Considering Flat Euclidean space in radial and setting

$$\hat{g}_{00} F^2\left(r, \theta\right) + B\Sigma\Delta_r^{-1} = 1, \tag{32}$$

we have

$$F\left(r, \theta\right) = \pm\sqrt{\left(1 - B\Sigma\Delta_r^{-1}\right) / \hat{g}_{00}}. \tag{33}$$

Considering Hawking tunneling radiation of uncharged particles occurred at the event horizon of Kaluza-Klein black hole, so Eq. (33) should be chosen the sign +. Then substituting Eq. (33) into Eq. (30), we obtain the Painlevé-Kaluza-Klein line element

$$\begin{aligned} ds^2 = {} & \hat{g}_{00} dt^2 + 2\sqrt{\hat{g}_{00}\left(1 - B\Sigma\Delta_r^{-1}\right)} dt dr + dr^2 + [\hat{g}_{00} G^2\left(r, \theta\right) + B\Sigma] d\theta^2 \\ & + 2\sqrt{\hat{g}_{00}\left(1 - B\Sigma\Delta_r^{-1}\right)} G\left(r, \theta\right) dr d\theta + 2\hat{g}_{00} G\left(r, \theta\right) dt d\theta. \end{aligned} \tag{34}$$

According to Landau's condition of coordinate clock synchronization (Landau and Lifshitz, 1975)

$$\frac{\partial}{\partial x^i}\left(-\frac{g_{0j}}{\hat{g}_{00}}\right) = \frac{\partial}{\partial x^j}\left(-\frac{g_{0i}}{\hat{g}_{00}}\right), \tag{35}$$

we have

$$\frac{\partial F(r,\theta)}{\partial \theta} = \frac{\partial G(r,\theta)}{\partial r}, \tag{36}$$

the condition is in accordance with Eq. (29), so the Painlevé-Kaluza-Klein coordinate system satisfy Landau's condition of coordinate synchronization. That is to say, we can define the coordinate clock synchronization in the Painlevé-Kaluza-Klein coordinate system.

The Painlevé-Kaluza-Klein coordinate system has a attractive features. First, coordinate singularity at the event horizon does not exist. Second, the infinite red-shift surface is in coincidence with the event horizon. Third, space-time is stationary. Fourth, constant-time slices are just flat Euclidean space in radial. Fifth, it satisfies Landau's condition of coordinate synchronization.

Considering the uncharged particle's radial motion and tunneling from the event horizon as an ellipsoid shell, the particle should be still an ellipsoid shell during the tunneling process to conserve the symmetry of the Kaluza-Klein space-time. So from Eq. (34), radial null geodesics equation are given as

$$\dot{r} = \frac{dr}{dt} = \frac{[\pm B\Sigma - \sqrt{B\Sigma\,(B\Sigma - \Delta_r)}]\sqrt{1-\nu^2}}{\sqrt{(r^2+a^2)^2 - \Delta_r(a^2\sin^2\theta + \nu^2\Sigma)}}, \tag{37}$$

where the sign $+$ corresponds to an outgoing geodesic and the sign $-$ corresponds to an ingoing geodesic respectively.

5. THE TUNNELING RADIATION CHARACTERISTICS

Now, Let's move on to discuss the tunneling radiation characteristics of Kaluza-Klein black hole. For the sake of simplicity, we only consider the tunneling radiation of uncharged particles. In our discussion, we can consider the picture of a pair of virtual particles spontaneously created inside the horizon, the positive energy virtual particle can tunnel out and the negative energy particle is absorbed by the black hole. Taking the particle's self-gravitation, energy conservation, angular momentum conservation, and the tunneling particle as a shell (an ellipsoid shell) of energy ω' and angular momentum $\omega' a$ into account, when the particle is tunneled out as an ellipsoid shell, fixing the total mass and angular momentum of the space-time and allowing those of the black hole to fluctuate, then the mass and the angular momentum of the black hole will be replaced by $(M-\omega')$ and $(M-\omega')a$ respectively. Meanwhile the event horizon will shrink, we refer to the cases pre- and post shrinking as two turning points of potential barrier, the distance between the two turning points is the width of potential barrier and decided by the energy of outgoing particle. At this critical moment, the mass parameter M in radial null geodesics (37) and the Painlevé-Kaluza-Klein line element (34) will be replaced by $(M-\omega')$ and the event horizon can be written as

$$r'_+ = \frac{2(1-\nu^2)}{2-\nu^2}(M-\omega') + \sqrt{\left[\frac{2(1-\nu^2)}{2-\nu^2}\right]^2 (M-\omega')^2 - a^2}, \tag{38}$$

and the dragging angular velocity at the event horizon of the black hole

$$\Omega'_+ = \frac{a\sqrt{1-\nu^2}}{(r'_+)^2+a^2} = \frac{a\sqrt{1-\nu^2}}{2\left[\frac{2(1-\nu^2)}{(2-\nu^2)}\right]^2 (M-\omega')^2 + 2\left[\frac{2(1-\nu^2)}{(2-\nu^2)}\right](M-\omega')\sqrt{\left[\frac{2(1-\nu^2)}{(2-\nu^2)}\right]^2 (M-\omega')^2 - a^2}}. \tag{39}$$

As the event horizon is in coincidence with the infinite red-shift surface, the geometrical optics limit is approximately reliable. According to the WKB approximation, the tunneling rate and the action of the particle satisfy (Kraus and Keski-Vakkuri, 1997; Kraus and Parentani, 2000)

$$\Gamma \sim e^{-2\operatorname{Im} S} \tag{40}$$

where S is the action of the particle, and

$$S = \int_{t_i}^{t_f} L\,(r, \dot{r}, \varphi, \dot{\varphi}, t)dt, \tag{41}$$

where $L\,(r, \dot{r}, \varphi, \dot{\varphi}, t)$ is the Lagrangian function. In Eq. (34), the coordinate φ is not existed, that is to say, φ is an ignorable coordinate in Lagrangian function. In order to eliminate the freedom of φ, the action of the particle can be expressed as

$$S = \int_{t_i}^{t_f} (L - p_\varphi \dot{\varphi})dt, \tag{42}$$

so the imaginary part of the action can be expressed as

$$\operatorname{Im} S = \operatorname{Im}\left[\int_{r_i}^{r_f} p_r dr - \int_{\varphi_i}^{\varphi_f} p_\varphi d\varphi\right] = \operatorname{Im}\left[\int_{r_i}^{r_f}\int_0^{p_r} dp'_r dr - \int_{r_i}^{r_f}\int_0^{p_\varphi} \frac{\dot{\varphi} dp'_\varphi}{\dot{r}} dr\right] \tag{43}$$

According to Hamilton equation, we have

$$\dot{r} = \left.\frac{dH}{dp_r}\right|_{(r;\varphi;p_\varphi)} = \frac{d(M-\omega')}{dp_r} = -\frac{d\omega'}{dp_r},$$
$$\dot{\varphi} = \left.\frac{dH}{dp_\varphi}\right|_{(r;\varphi;p_r)} = a\Omega'_+ \frac{d\left(M-\omega'\right)}{dp_\varphi} = -a\Omega'_+ \frac{d\omega'}{dp_\varphi}. \tag{44}$$

Substituting Eq. (44) into Eq. (43), we have

$$\operatorname{Im} S = \operatorname{Im}\left[\int_M^{M-\omega}\int_{r_i}^{r_f}\frac{dr}{\dot{r}}d\left(M-\omega'\right) - \int_M^{M-\omega}\int_{r_i}^{r_f} a\Omega'_+ \frac{dr}{\dot{r}}d\left(M-\omega'\right)\right]. \tag{45}$$

Considering Eq. (37), and noting that we must replace M with $M-\omega'$ and choose the $+$ sign, we obtain

$$\operatorname{Im} S = \operatorname{Im}\left[\int_M^{M-\omega}\int_{r_i}^{r_f}(1-a\Omega'_+)\frac{\sqrt{(r^2+a^2)^2-\Delta'_r(a^2\sin^2\theta+\nu^2\Sigma)}}{[B\Sigma-\sqrt{B\Sigma(B\Sigma-\Delta'_r)}]\sqrt{1-\nu^2}}drd(M-\omega')\right], \tag{46}$$

where

$$\Delta'_r = r^2+a^2-2\left[\frac{2(1-\nu^2)}{2-\nu^2}\right](M-\omega') = (r-r'_+)(r-r'_-), \tag{47}$$

$$r'_- = \frac{2(1-\nu^2)}{2-\nu^2}(M-\omega') - \sqrt{\left[\frac{2(1-\nu^2)}{2-\nu^2}\right]^2(M-\omega')^2-a^2}, \tag{48}$$

$$r_i = \frac{2(1-\nu^2)}{2-\nu^2}M + \sqrt{\left[\frac{2(1-\nu^2)}{2-\nu^2}\right]^2 M^2-a^2}, \tag{49}$$

$$r_f = \frac{2(1-\nu^2)}{2-\nu^2}(M-\omega) + \sqrt{\left[\frac{2(1-\nu^2)}{2-\nu^2}\right]^2(M-\omega)^2-a^2}. \tag{50}$$

We multiply and divide the integrand with $B\Sigma + \sqrt{B\Sigma(B\Sigma-\Delta'_r)}$ to obtain

$$\operatorname{Im} S = \operatorname{Im}\left[\int_M^{M-\omega}\int_{r_i}^{r_f}(1-a\Omega'_+)\frac{\sqrt{(r^2+a^2)^2-\Delta'_r(a^2\sin^2\theta+\nu^2\Sigma)}\left[B\Sigma+\sqrt{B\Sigma(B\Sigma-\Delta'_r)}\right]}{B\Sigma\sqrt{1-\nu^2}(r-r'_+)(r-r'_-)}drd(M-\omega')\right] \tag{51}$$

Obviously, in the vicinity of the event horizon, $r = r'_+$ is a pole. Doing the r integral first we have

$$\operatorname{Im} S = \int_M^{M-\omega} -\frac{2\pi}{\sqrt{1-\nu^2}} \left\{ \sqrt{\left[\frac{2(1-\nu^2)}{2-\nu^2}\right]^2 (M-\omega')^2 - a^2} + \left[\frac{2(1-\nu^2)}{2-\nu^2}\right](M-\omega') + \frac{a^2/2}{\sqrt{\left[\frac{2(1-\nu^2)}{2-\nu^2}\right](M-\omega')^2 - a^2}} + \frac{a^2/2(1-\sqrt{1-\nu^2})}{\sqrt{\left[\frac{2(1-\nu^2)}{2-\nu^2}\right](M-\omega')^2 - a^2}} \right\} d(M-\omega'). \tag{52}$$

Finishing the integration, and carrying on integrality on $(M-\omega')$, we obtain

$$\begin{aligned} \operatorname{Im} S = \frac{\pi}{\sqrt{1-\nu^2}} \left[\frac{2(1-\nu^2)}{2-\nu^2}\right]^2 \Bigg\{ & M^2 - (M-\omega)^2 + \left[\frac{(2-\nu^2)}{2(1-\nu^2)}\right] M \sqrt{\left[\frac{2(1-\nu^2)}{2-\nu^2}\right]^2 M^2 - a^2} \\ & - \left[\frac{(2-\nu^2)}{2(1-\nu^2)}\right](M-\omega)\sqrt{\left[\frac{2(1-\nu^2)}{2-\nu^2}\right]^2 (M-\omega)^2 - a^2} - a^2(1-\sqrt{1-\nu^2}) \\ & \times \ln \frac{\left[2(1-\nu^2)/(2-\nu^2)\right](M-\omega) + \sqrt{\left[2(1-\nu^2)/(2-\nu^2)\right]^2 (M-\omega)^2 - a^2}}{\left[2(1-\nu^2)/(2-\nu^2)\right] M + \sqrt{\left[2(1-\nu^2)/(2-\nu^2)\right]^2 M^2 - a^2}} \Bigg\}. \end{aligned} \tag{53}$$

Contrasted to the total mass of the black hole M, the energy of the emission particle ω can be omitted, when α is a real, and is given as follows

$$\alpha = \left\{ \left[\frac{\left[2(1-\nu^2)/(2-\nu^2)\right](M-\omega) + \sqrt{\left[2(1-\nu^2)/(2-\nu^2)\right]^2 (M-\omega)^2 - a^2}}{\left[2(1-\nu^2)/(2-\nu^2)\right] M + \sqrt{\left[2(1-\nu^2)/(2-\nu^2)\right]^2 M^2 - a^2}} \right]^{\frac{2\pi a^2(1-\sqrt{1-\nu^2})}{\sqrt{1-\nu^2}} \left[\frac{2(1-\nu^2)}{2-\nu^2}\right]^2} \right\}^{-1} \tag{54}$$

we can learn that $\alpha \sim 1$. So the relationship between the tunneling rate and the action of the particle satisfies

$$\Gamma \sim \exp(-2\operatorname{Im} S) \sim \alpha \exp(-2\operatorname{Im} S), \tag{55}$$

The tunneling rate of outgoing particles is

$$\Gamma \sim \exp\left\{\frac{-2\pi}{\sqrt{1-\nu^2}}\left[\frac{2(1-\nu^2)}{2-\nu^2}\right]^2\left(M^2-(M-\omega)^2+\left[\frac{(2-\nu^2)}{2(1-\nu^2)}\right]\right.\right.$$
$$\times M\sqrt{\left[\frac{2(1-\nu^2)}{2-\nu^2}\right]^2 M^2-a^2}-\left[\frac{(2-\nu^2)}{2(1-\nu^2)}\right](M-\omega)$$
$$\left.\left.\times\sqrt{\left[\frac{2(1-\nu^2)}{2-\nu^2}\right]^2(M-\omega)^2-a^2}\right)\right\}=\exp\left(\frac{A'_+}{4}-\frac{A_+}{4}\right)$$
$$=\exp[S_{\rm BH}(M-\omega)-S_{\rm BH}(M)]=\exp(\Delta S_{\rm BH}). \quad (56)$$

where $S_{\rm BH}$ is Bekenstein-Hawking entropy of stationary Kaluza-Klein black hole, $\Delta S_{\rm BH}=S_{\rm BH}(M-\omega)-S_{\rm BH}(M)$ is the difference of the entropies of the black hole before and after the emission. Thus the tunneling spectrum of Kaluza-Klein black hole is not pure thermal. This result obviously consists with an underlying unitary theory and is a good correction to Hawking pure thermal spectrum.

6. DISCUSSION

When $\nu = 0$, stationary Kaluza-Klein black hole will be reduced to stationary Kerr black black hole. Taking self-gravitation action, energy conservation and angular momentum conservation into account, the event horizon responding to the cases pre- and post shrinking are given by

$$r_+^K = M+\sqrt{M^2-a^2},\quad r'^K_+ = (M-\omega)+\sqrt{(M-\omega)^2-a^2}, \quad (57)$$

accordingly, the areas of the black hole are

$$A_+^K = 4\pi[(r_+^K)^2+a^2] = 8\pi(M^2+M\sqrt{M^2-a^2}),$$
$$A'^K_+ = 4\pi[(r'^K_+)^2+a^2] = 8\pi[(M-\omega)^2+(M-\omega)\sqrt{(M-\omega)^2-a^2}]. \quad (58)$$

From Eq. (26), we can get Hawking pure thermal spectrum

$$N_\omega = \frac{1}{e^{(\omega-\omega_0)/T_K}-1}, \quad (59)$$

where

$$\omega_0 = \varsigma\Omega_+^K = \frac{\varsigma a}{2M^2+2M\sqrt{M^2-a^2}}, \quad (60)$$

$$T_K = \frac{1}{4\pi}\frac{\sqrt{M^2-a^2}}{M^2+M\sqrt{M^2-a^2}}, \quad (61)$$

According to the derived tunneling rate across the event horizon of Kaluza-Klein black hole, we can obtain that of Kerr black hole

$$\Gamma_K \sim e^{-2\pi[M^2-(M-\omega)^2+M\sqrt{M^2-a^2}-(M-\omega)\sqrt{(M-\omega)^2-a^2}]}$$
$$= e^{(\frac{A'^K_+}{4}-\frac{A^K_+}{4})} = e^{[S^K_{\mathrm{BH}}(M-\omega)-S^K_{\mathrm{BH}}(M)]} = e^{\Delta S^K_{\mathrm{BH}}}, \quad (62)$$

where S^K_{BH} is Bekenstein-Hawking entropy of stationary Kerr black hole. According to Zhang and Zhao (2005), expand ΔS^K_{BH} in $(\omega-\omega_0)$ and neglect the higher-order term, and we have

$$\Gamma_K \sim e^{\Delta S_{\mathrm{BH}}} = e^{-\frac{1}{T_K}(\omega-\omega_0)[1-\frac{(r^K_+)^2+a^2}{(r^K_+)^4}(M+\sqrt{M^2-a^2}-\frac{Ma^2}{2(M^2-a^2)})(\omega-\omega_0)]}. \quad (63)$$

From Eqs. (59) and (63), we find that ignoring the spectrum at higher energies the tunneling radiation spectrum is coincident with Hawking pure thermal one. So the tunneling radiation spectrum carries on a correction to Hawking pure thermal one, and is actually exact.

When $l=0$ and $a=0$, it is static Swarzschild black hole. From Eq. (26), Hawking pure thermal spectrum of the black hole is given as follows

$$N_\omega = \frac{1}{e^{\omega/T_s}-1}, \quad (64)$$

where

$$T_s = \frac{1}{8\pi M}. \quad (65)$$

From Eq. (56), the tunneling rate of static Swarzschild black hole can be expressed as

$$\Gamma_s \sim e^{-8\pi M\omega(1-\frac{\omega}{2M})} = e^{\Delta S^s_{\mathrm{BH}}}. \quad (66)$$

Through Eq. (66), we can learn that the leading term of the tunneling rate give the Hawking radiation spectrum. The second term is the correction from the response of the background geometry to emission. So, when the self-gravitation, energy conservation and angular momentum conservation are taken into consideration, the tunneling rate at the event horizon of the black hole is relevant to the change of Bekenstein-Hawking entropy and the derived radiation spectrum deviates from pure thermal. So further research indicates that the tunneling radiation spectrum gives a correction to Hawking pure thermal one, and is actually exact.

Taking all the above research into account, the black hole radiation causes the space-time background geometry varied, the event horizon of the black hole also changes with the radiation as result of the self-gravitation, energy conservation and angular momentum conservation. In other words, when the particle tunnel out, the event horizon will contract and the two turning points pre- and post emission are the two points of barrier. The result shows that the tunneling rate at the event horizon of the black hole is relevant to the change of Bekenstein- Hawking entropy and the derived radiation spectrum deviates from pure thermal.

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